Exact Values for the $\varepsilon$-Ascent Chromatic Index of Complete Graphs

C. M. van Bommel * J. Gorzny
Department of Mathematics and Statistics
University of Victoria, P.O. Box 1700 STN CSC
Victoria, BC, Canada V8W 2Y2
{cvanbomm,jgorzny}@uvic.ca
July 16, 2014

Abstract
Following a problem introduced by Schurch [M. Schurch, On the Depression of Graphs, Doctoral Dissertation, University of Victoria, 2013], we find exact values of the minimum number of colours required to properly edge colour $K_n$, $n \geq 6$, using natural numbers, such that the length of a shortest maximal path of increasing edge labels is equal to three. This result improves the result of Breytenbach and Mynhardt [A. Breytenbach and C. M. Mynhardt, On the $\varepsilon$-Ascent Chromatic Index of Complete Graphs, Involve, to appear].

1 Introduction
An edge ordering of a graph $G = (V, E)$ is an injection $f : E \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A path $v_1, e_1, \ldots, e_{k-1}, v_k$ (with $v_1 \neq v_k$) in $G$ for which the edge ordering $f(e_1) < \cdots < f(e_{k-1})$ increases along its edge sequence is called an $f$-ascent; an $f$-ascent is maximal if it is not contained in a longer $f$-ascent. The flatness of $f$, denoted $h(f)$, is the length of a shortest maximal ascent. The depression $\varepsilon(G)$ of $G$ is the smallest integer $k$ such that any edge ordering $f$ has a maximal $f$-ascent of length at most $k$.

An edge ordering for a graph $G$ is also a proper edge colouring: no two adjacent edges have the same label. The minimum number of labels, or colours, in a proper edge colouring is called the edge chromatic number or the chromatic index $\chi'(G)$. The minimum number of colours in a proper

*Research supported by NSERC.
edge colouring $c$ such that $h(c) = \varepsilon(G)$ is called the $\varepsilon$-ascent chromatic index of $G$ and is denoted by $\chi_\varepsilon(G)$.

As shown in [3], $\varepsilon(K_n) = 3$ for all $n \geq 4$. This fact prompted Schurch [4, 5] to introduce the following problem, where $r(n)$ is the same as $\chi_\varepsilon(K_n)$:

**Question 1.** For $n \geq 4$, what is the smallest integer $r(n)$ for which there exists a proper edge colouring of $K_n$ in colours $1, \ldots, r(n)$ such that a shortest maximal path of increasing edge labels has length three?

Schurch [4, 5] showed that $r(n) \leq 2n - 3$ for all $n \geq 4$, which allowed him to determine $r(n)$ for $n \in \{4, 5\}$ as well as bound the value of $r(6)$. In [1], Breytenbach and Mynhardt provided a lower bound for $r(n) = \chi_\varepsilon(K_n)$:

**Theorem 2.** If $n \geq 4$, then

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod{4} \\ n + 1 & \text{if } n \equiv 1, 2 \pmod{4} \\ n + 2 & \text{if } n \equiv 3 \pmod{4} \end{cases}.$$  

Further, they improved the general upper bound to $r(n) \leq \left\lceil \frac{3n - 3}{2} \right\rceil$. For even $n$, they provided better bounds: in the case $n \equiv 2 \pmod{4}$ they show that $r(n) = n + 1$, and in the case $n \equiv 0 \pmod{4}$ they show that $n \leq r(n) \leq n + 1$. Using these bounds, they also achieve $r(7) = 9$.

Breytenbach and Mynhardt conclude with the following conjecture:

**Conjecture 3.** For all $n \geq 4$, $\chi_\varepsilon(K_n) = \chi'(K_n) + 2$.

Since it is well known (see, e.g. [2], Section 10.2) that $\chi'(K_{2n}) = 2n - 1$ and $\chi'(K_{2n+1}) = 2n + 1$, a colouring of $K_8$ that illustrates $\chi_\varepsilon(K_8) \leq 8$ is a counter-example to the conjecture, and similarly a colouring of $K_9$ that illustrates $\chi_\varepsilon(K_9) \leq 10$ is another. For $K_8$ such a colouring is contained in the proof for the case $n \equiv 0 \pmod{4}$ in Section 2.1, for $K_9$, such a colouring is provided in Appendix B as Figure 5; it was verified by computer that this colouring has flatness equal to three. Motivated by these counter-examples, we determine the exact values of $\chi_\varepsilon$ for all $n \geq 4$.

## 2 Improved Upper Bounds

For $n \geq 6$, we show the existence of proper edge colourings with flatness equal to three and with the number of colours equal to the lower bound on $\chi_\varepsilon(K_n)$ established by Breytenbach and Mynhardt [1], and thus complete the computation of the exact value of $\chi_\varepsilon(K_n)$ for all $n$. The value was determined exactly for $n \leq 5$ in [4, 5] and for $n = 7$ and $n \equiv 2 \pmod{4}$.
in [1]. We consider \( n \equiv 0 \pmod{4} \) in Section 2.1, \( n \equiv 1 \pmod{4} \) in Section 2.2, and \( n \equiv 3 \pmod{4} \) in Section 2.3. In each case, we make use of the following fact [1].

**Fact 4.** To prove that \( h(c) = 3 \), where \( c \) is a proper edge colouring of \( K_n \), it is sufficient to prove the following statement:

- **S:** For any \( y \in V(K_n) \) and edges \( e = xy \) and \( f = yz \) such that \( c(e) < c(f) \), there exists
  - (a) an edge \( tx, t \notin \{x, y, z\} \), such that \( c(tx) < c(e) \), or
  - (b) an edge \( tz, t \notin \{x, y, z\} \), such that \( c(tz) < c(f) \).

### 2.1 The case \( n \equiv 0 \pmod{4} \)

Say \( n = 4m, m \geq 2 \), and \( V(K_n) = \{v_0, \ldots, v_{4m-1}\} \). Let \( G \) and \( H \) be the subgraphs of \( K_n \) induced by \( \{v_0, \ldots, v_{2m-1}\} \) and \( \{v_{2m}, \ldots, v_{4m-1}\} \), respectively. Then \( G \cong H \cong K_{2m} \) and each of them is \((2m - 1)\)-edge colourable. We describe a colouring \( c \) of \( K_n \) in the colours \( 1, \ldots, 4m \) as follows.

- In \( G \), let \( c \) be any proper edge colouring of \( K_{2m} \) in the \( 2m - 1 \) colours \( \{1\} \cup \{m + 2, \ldots, 3m - 1\} \).
- In \( H \), let \( c \) be any proper edge colouring of \( K_{2m} \) in the \( 2m - 1 \) colours \( \{4m\} \cup \{m + 2, \ldots, 3m - 1\} \).
- We still need to colour the edges of the complete bipartite graph \( F \cong K_{2m, 2m} \) induced by the edges \( v_iv_j, i \in \{0, \ldots, 2m - 1\}, j \in \{2m, \ldots, 4m - 1\} \). But \( \chi'(K_{2m, 2m}) = 2m \) and there are \( 2m \) unused colours \( 2, \ldots, m + 1 \) and \( 3m, \ldots, 4m - 1 \). Colour the edges of \( F \) with these colours in such a way that the graph induced by the edges assigned the colours \( 1, 2, 4m - 1, 4m \) is triangle free. This can be achieved by partitioning the vertices of \( K_n \) into sets \( X \) and \( Y \) such that each edge labelled with \( 1, 2, 4m - 1, \) and \( 4m \) has an end in \( X \) and an end in \( Y \), which is possible as \( m \geq 2 \). Then the graph induced by the edges assigned the colours \( 1, 2, 4m - 1, 4m \) is bipartite, and hence triangle free.

As an example, a colouring of \( K_8 \) is given in Figure 1. It is clear that \( c \) is a proper edge colouring of \( K_{4m} \) in \( 4m \) colours.

**Theorem 5.** For all \( m \geq 2 \), the colouring \( c \) of \( K_{4m} \) has flatness equal to three.

**Proof.** Let \( F, G, \) and \( H \) be the subgraphs of \( K_{4m} \) defined above and let \( e, f \in E(K_{4m}) \) be adjacent edges such that \( c(e) < c(f) \). By Fact 4, it is
Figure 1: Edge colouring $c$ of $K_8$

sufficient to show $S(a)$ or $S(b)$ holds. Let $e = v_jv_i$ and $f = v_iv_k$. Observe at each vertex, every colour is used at an incident edge, except exactly one of 1 or $4m$ is used. Thus if $c(e) \geq 4$, at least one of 2, 3 is not used to colour edge $v_jv_k$, and thus $v_j$ is adjacent to some vertex $v_l$, $l \neq k$, such that $c(v_jv_k) = c(e)$, so $S(a)$ holds. Similarly, if $c(f) \leq 4m - 3$, at least one of $4m - 2, 4m - 1$ is not used to colour edge $v_kv_j$, and thus $v_k$ is adjacent to some vertex $v_l$, $l \neq j$, such that $c(v_kv_l) = c(f)$, so $S(b)$ holds. Thus we consider $c(e) \in \{1, 2, 3\}$ and $c(f) \in \{4m - 2, 4m - 1, 4m\}$.

Suppose $c(e) = 1$. By construction, $c(f) \neq 4m$. Thus $c(e) \in \{4m - 2, 4m - 1\}$, so $e \in E(G)$ and $f \in E(F)$. Thus there exists an edge $v_kv_l$ such that $c(v_kv_l) = 4m$. As $j \neq l$, $S(b)$ holds. Similarly, if $c(f) = 4m$, $c(e) \in \{2, 3\}$, so $f \in E(H)$ and $e \in E(F)$. Thus there exists an edge $v_kv_j$ such that $c(v_kv_j) = 1$. As $j \neq l$, $S(a)$ holds.

Now we consider $c(e) \in \{2, 3\}$ and $c(f) \in \{4m - 2, 4m - 1\}$, and thus $e, f \in E(F)$. If $c(e) = 3$, then there exists an edge $v_kv_j \in E(F)$ such that $c(v_kv_j) = 2$ and $l \neq k$ as $F$ is bipartite. Thus $S(a)$ holds. Similarly, if $c(f) = 4m - 2$, then there exists an edge $v_kv_l \in E(F)$ such that $c(v_kv_l) = 4m - 1$ and $l \neq k$ as $F$ is bipartite. Thus $S(b)$ holds.

Finally, we consider $c(e) = 2$ and $c(f) = 4m - 1$. If $v_i \in V(G)$, then there exists an edge $v_kv_l \in E(H)$ such that $c(v_kv_l) = 4m$ and $l \neq j$ by
construction, so $S(b)$ holds. If $v_i \in V(H)$, then there exists an edge $v_iv_j \in E(G)$ such that $c(v_iv_j) = 1$ and $i \neq k$ by construction, so $S(a)$ holds. □

Thus we conclude the following.

**Corollary 6.** For all $n \geq 8$ and $n \equiv 0 \pmod{4}$, $\chi_e(K_n) = n$.

### 2.2 The case $n \equiv 1 \pmod{4}$

Say $n = 4m + 1$, $m \geq 3$, and $V(K_n) = \{u_0, \ldots, u_{2m-1}, v_0, \ldots, v_{2m-1}, w\}$. Let $G$ and $H$ be the subgraphs of $K_n$ induced by $\{u_0, \ldots, u_{2m-1}\}$ and $\{v_0, \ldots, v_{2m-1}\}$, respectively. Then $G \cong H \cong K_{2m}$ and each of them is $(2m-1)$-edge colourable. Let $F$ be the subgraph of $K_n$ induced by the edges $u_iv_j$, $0 \leq i, j \leq 2m-1$. Then $F \cong K_{2m,2m}$ and is $2m$-edge colourable. Let $c$ be a colouring of $K_n - w$ with the following colour classes:

**F:** For $0 \leq k \leq 2m-1$, let $E^F_k = \{u_iv_{i+k} : 0 \leq i \leq 2m-1\}$, indices taken mod $2m$.

**G** and **H:** Let $\{a_j\}$, $0 \leq j \leq 2m-2$, be the sequence $2m-2$, $2m-4$, $\ldots$, 2, 1, 3, $\ldots$, $2m-1$. For $0 \leq k \leq 2m-2$, let $E^G_k = \{u_0u_{a_k}\} \cup \{u_{a_k-i}u_{a_{k+i}} : 1 \leq i \leq m-1\}$ and $E^H_k = \{v_0v_{a_k}\} \cup \{v_{a_{k-i}}v_{a_{k+i}} : 1 \leq i \leq m-1\}$, indices taken mod $2m-1$.

Later we will pair the colour classes of $G$ and $H$ to get exactly $4m-1$ colours. We form the colouring $c^*$ of $K_n$ as follows. Assume $c$ uses the colours $1, \ldots, 2m, 2m+3, \ldots, 4m+2$. Define the path $P$ as follows: if $m \equiv 1 \pmod{2}$, $P = u_0, u_2, \ldots, u_{2m-2}, v, v_1, u_1, v_3, u_3, u_{2m-1}, v_5, u_5, v_{m+1}, v_{m+2}, u_{m+2}, v_{m+4}, u_{m+4}, v_{m+6}, \ldots, v_{m-1}, u_{m+1}, v_2, u_m, v_4, \ldots, u_3, v_{m+1}, v_{m+3}, \ldots, v_{2m-2}, v_0, v_{2m-1}, v_{2m-3}, \ldots, v_{m+2}$, and if $m \equiv 0 \pmod{2}$, $P = v_1, u_{2m-1}, v_3, u_{2m-3}, \ldots, v_{m-1}, u_{m+1}, v_2, u_m, v_4, u_{m-3}, \ldots, v_m, u_1, u_2, v_4, \ldots, u_2, v_{2m-2}, v_{2m-2}, v_{2m-4}, \ldots, v_{m+1}, v_{2m-1}, v_{2m-3}, \ldots, v_{m+1}, w, u_0$. For small values of $m$, the path $P$ is shown in Figure 2. For each edge $xy$ of $P$, if $xy$ occurs before $w$ in the path, let $c^*(xw) = c(xy)$, otherwise, if $xy$ occurs after $w$ in the path, let $c^*(yw) = c(xy)$. Then if the edges on the path are enumerated, let $c^*(xy) = 2m+1$ if $xy$ is an odd edge, otherwise let $c^*(xy) = 2m+2$ if $xy$ is an even edge. Finally, if $e \in E(K_n - w)$ is not in $P$, let $c^*(e) = c(e)$.

For $c^*$ to be a proper colouring of $K_n$, each edge of $P$ must belong to a different colour class in $c$. We prove this statement in the following claim.

**Claim 7.** Each edge in $P$ that does not have $w$ as an endpoint belongs to a different colour class.

**Proof.** We consider each of the two paths separately.
Figure 2: The path $P$ for small values of $m$
$m \equiv 1 \pmod{2}$: The first $m - 1$ edges are contained in $G$. The edge $u_0u_2$ is in colour class $E_{m-1}^G$ and each edge of the form $u_{2j}u_{2(j+1)}$, $1 \leq j \leq m - 2$ is in colour class $E_{2m-j-2}^G$. The next two edges are incident with $w$. The following $2m$ edges are contained in $F$. Each edge of the form $v_{2j-1}v_{2m-2j+3}$, $1 \leq j \leq \frac{m+1}{2}$ is in the colour class $E_{4j-4}^F$, each edge of the form $u_{2m-2j+3}v_{2j+1}$, $1 \leq j \leq \frac{m-1}{2}$ is in the colour class $E_{4j-2}^F$, each edge of the form $u_{m-2j+4}v_{2j}$, $1 \leq j \leq \frac{m+1}{2}$ is in the colour class $E_{4j+m-4}^F$, and each edge of the from $v_{2j}u_{m-2j+2}$, $1 \leq j \leq \frac{m-1}{2}$ is in the colour class $E_{4j+m-2}^F$. The final $m - 1$ edges are contained in $H$. Each edge of the form $v_{m+2j-1}v_{m+2j+1}$, $1 \leq j \leq \frac{m-3}{2}$ is in the colour class $E_{2m-2j+3}^H$, the edge $v_{2m-2m}v_0$ is in the colour class $E_0^H$, the edge $v_0v_{2m-1}$ is in the colour class $E_{2m-2}^H$, and each edge of the form $v_{2m-2j+1}v_{2m-2j-1}$, $1 \leq j \leq \frac{m-3}{2}$ is in the colour class $E_{m-1-j}^H$. No two edges taken from $F$ belong to the same colour class as the first half are consecutive even numbered colour classes, and the second half are consecutive odd numbered colour classes, mod $2m$, starting with $m$. No two edges taken from $G$ belong to the same colour class as $m - 1 = 2m - j - 2 \implies j = m - 1$. Finally, no two edges taken from $H$ belong to the same colour class as $0 < \frac{m+1}{2}$, $m - 2 < m$, and $\frac{3m-5}{2} < 2m - 2$ for positive $m$.

$m \equiv 0 \pmod{2}$: The first $2m - 1$ edges are contained in $F$. Each edge of the form $v_{2j}v_{2m-2j+1}$, $1 \leq j \leq \frac{m}{2}$ is in the colour class $E_{4j-2}^F$, each edge of the form $u_{2m-2j+1}v_{2j+1}$, $1 \leq j \leq \frac{m-2}{2}$ is in the colour class $E_{4j}^F$, each edge of the form $u_{m-2j+3}v_{2j}$, $1 \leq j \leq \frac{m}{2}$ is in the colour class $E_{4j-m-3}^F$, and each edge of the form $v_{2j}u_{m-2j+1}$, $1 \leq j \leq \frac{m}{2}$ is in the colour class $E_{4j-m-1}^F$. The next $m - 1$ edges are contained in $G$. The edge $u_1u_2$ is in the colour class $E_{2m-2}^G$ and each edge of the form $u_{2j}u_{2(j+1)}$, $1 \leq j \leq m - 2$ is in colour class $E_{2m-j-2}^G$. The following edge, $u_{2m-2}v_{2m-2}$ is contained in $F$ and is in colour class $E_0^F$. The next $m - 1$ edges are contained in $H$. Each edge of the form $v_{2m-2j}v_{2m-2j-2}$, $1 \leq j \leq \frac{m-2}{2}$ is in the colour class $E_{m+j-2}^H$, the edge $v_{m+2}v_0$ is in the colour class $E_{m-1-j}^H$, the edge $v_0v_{2m-1}$ is in the colour class $E_{2m-2}^H$, and each edge of the form $v_{2m-2j+1}v_{2m-2j-1}$, $1 \leq j \leq \frac{m-2}{2}$ is in the colour class $E_{m-1-j}^H$. The final two edges are incident with $w$.

No two edges taken from $F$ belong to the same colour class as the first half are consecutive even numbered colour classes starting from 2, the second half are consecutive odd numbered colour classes, mod $2m$, starting with $1 - m \equiv m + 1$, and the final edge is in the colour class 0. No two edges taken from $G$ belong to the same colour class as
\[2m - 2 = 2m - j - 2 \Rightarrow j = 0.\] Finally, no two edges taken from \(H\) belong to the same colour class as \(\frac{m - 2}{2} < \frac{m}{2}, \ m - 2 < m - 1,\) and \(\frac{3m - 8}{2} < 2m - 2\) for positive \(m.\)

Therefore, \(c^*\) is a proper colouring of \(K_n.\) It remains to show that there is a colouring \(c\) which, when extended to \(c^*,\) allows us to avoid maximal 2-ascents. We assign the colours to the colour classes in the following manner.

- Let \(E_{m-1}^G\) be assigned colour 1 and \(E_{m-1}^H\) be assigned colour \(4m + 2.\)

- If \(m \equiv 0, 1 \pmod{4},\) let \(E_0^F\) be assigned colour 2, \(E_1^F\) be assigned colour 3, \(E_2^F\) be assigned colour 4, and \(E_3^F\) be assigned colour \(4m + 1.\)

If \(m \equiv 2, 3 \pmod{4},\) let \(E_{2m-1}^F\) be assigned colour 2, \(E_0^F\) be assigned colour 3, \(E_1^F\) be assigned colour 4, and \(E_2^F\) be assigned colour \(4m + 1.\)

As a result, when \(c\) is extended to \(c^*,\) the edges incident with \(w\) assigned 2 and 3 have their other endpoint in \(V(G),\) and the edges incident with \(w\) assigned 4 and \(4m + 1\) have their other endpoint in \(V(H).\) Assign the remaining colour classes of \(F\) from the colours \(\{4, \ldots, m + 1, 3m + 2, \ldots, 4m - 1\}.\)

- For the remaining colour classes of \(G\) and \(H,\) assign from the colours \(\{m + 2, \ldots, 2m, 2m + 3, \ldots, 3m + 1\}\) such that each colour class with an edge in \(P\) is assigned a different colour.

As an example, a colouring of \(K_{13}\) is given in Figure 4 in Appendix A. Note that this proof cannot be applied to \(K_9.\) In place of a proof of this small case, we used a computer to search\(^1\) for a 10-colouring of \(K_9\) with flatness three, and a result is shown in Figure 5 in Appendix B.

**Theorem 8.** For all \(m \geq 3,\) the colouring \(c^*\) of \(K_{4m+1}\) has flatness equal to three.

**Proof.** Let \(F, G,\) and \(H\) be the subgraphs of \(K_{4m+1}\) defined above, let \(W\) be the subgraph induced by the edges incident with \(w,\) and let \(e, f \in E(K_{4m+1})\) be adjacent edges such that \(c^*(e) < c^*(f).\) By Fact 4, it is sufficient to show \(S(a)\) or \(S(b)\) holds. Let \(e = xy\) and \(f = yz.\) Observe that at each vertex, exactly two colours are not incident with it, at least one of which is either 1 or \(4m + 2.\) Thus if \(c^*(e) \geq 5,\) at least two of 2, 3, 4 are incident with \(x,\) and thus \(x\) is adjacent to some \(t \neq z\) such that \(c^*(tx) < c^*(e),\) so \(S(a)\) holds. Similarly, if \(c^*(f) \leq 4m - 2,\) at least two of \(4m - 1, 4m, 4m + 1\) are incident with \(z,\) and thus \(z\) is adjacent to some \(t \neq x\) such that \(c^*(f) < c^*(zt),\) so \(S(b)\) holds. Thus we consider \(c^*(e) \in \{1, 2, 3, 4\}\) and \(c^*(f) \in \{4m - 1, 4m, 4m + 1, 4m + 2\}.

---

\(^1\)The code is available at: [http://www.math.uvic.ca/~jgorzny/ascent/](http://www.math.uvic.ca/~jgorzny/ascent/)
Suppose $c^*(e) = 1$. By construction, $c^*(f) \neq 4m + 2$. Thus $c^*(f) \in \{4m - 1, 4m, 4m + 1\}$, so $e \in E(G)$ and $f \in E(F) \cup E(W)$. If $f \in E(W)$, then $c^*(f) = 4m - 1$ and there exists an edge $wt$ such that $c^*(wt) = 4m + 1$. Otherwise, $f \in E(F)$, and there exists an edge $zt$ such that $c^*(zt) = 4m + 2$.

As $t \neq x$ in either case, $S(b)$ holds. Similarly, if $c^*(f) = 4m + 2$, $c^*(e) \in \{2, 3, 4\}$, so $f \in E(H)$ and $e \in E(F) \cup E(W)$. If $e \in E(W)$, then $c^*(e) = 4$, and there exists an edge $wt$ such that $c^*(wt) = 2$, otherwise, $e \in E(F)$, and there exists an edge $tx$ such that $c^*(tx) = 1$. As $t \neq z$ in either case, $S(a)$ holds.

Now we consider $c^*(e) \in \{2, 3, 4\}$ and $c^*(f) \in \{4m - 1, 4m, 4m + 1\}$, and thus $e, f \in E(F) \cup E(W)$. If $c^*(e) = 4$ and $x$ is incident with both colours 2 and 3, it is clear that $S(a)$ holds. If $x$ is incident with only one of the colours 2 and 3, then $x \in V(H)$. If $y = w$, then $c^*(yz) = 4m - 1$, and $z$ is incident with at least one of the colours $4m, 4m + 1$. Therefore, at least one of these four colours is incident with $x$ or $z$ but not assigned to $xz$, so there exists some $t$ such that either $t \neq z$ and $c^*(tx) < c^*(e)$ or $t \neq x$ and $c^*(f) < c^*(zt)$, so either $S(a)$ or $S(b)$ holds. If $y \neq w$ then $y \in V(G)$ and $z \in V(H) \cup \{w\}$. Thus there exists an edge $tx \in E(F)$ such that $c^*(tx) \in \{2, 3\}$ and $t \neq z$ as $t \in V(G)$; hence $S(a)$ holds. Similarly, if $c^*(f) = 4m - 1$, then clearly $S(b)$ holds unless $z$ is incident with only one of the colours $4m, 4m + 1$, in which case $z \in V(G)$. We have shown $S(a)$ or $S(b)$ holds if $c^*(e) = 4$, thus $y \neq w$. Hence, $y \in V(H)$ and $x \in V(G)$, and there exists an edge $zt \in E(F)$ such that $c^*(zt) \in \{4m, 4m + 1\}$ and $t \neq x$ as $t \in V(H)$ so $S(b)$ holds.

We now consider $c^*(e) \in \{2, 3\}$ and $c^*(f) \in \{4m, 4m + 1\}$. If $y = w$, then $x \in V(G)$, $z \in V(H)$, and there is an edge $tx \in E(G)$ such that $c^*(tx) = 1$ and as $t \neq z$, $S(a)$ holds. If $x = w$, then $y \in V(G)$, $z \in V(H)$, and there is an edge $zt \in E(H)$ such that $c^*(zt) = 4m + 2$ and as $t \neq x$, $S(b)$ holds. Similarly, if $z = w$, then $y \in V(H)$, $x \in V(G)$, and there is an edge $tx \in E(G)$ such that $c^*(tx) = 1$ and as $t \neq z$, $S(a)$ holds.

Otherwise, $e, f \in E(F)$. Suppose $c^*(e) = 2$. If $x, z \in V(G)$, let $x = u_i$. Either $z = u_{i-2}$ or $z = u_{i-3}$, and as either $c^*(u_{i-1}u_i) = 1$ or $c^*(u_{i+1}u_i) = 1$, then $S(a)$ holds. Otherwise, if $x, z \in V(H)$, let $z = v_i$. Either $x = v_{i-2}$ or $x = v_{i-3}$, and as either $c^*(v_{i-1}v_i) = 4m + 2$ or $c^*(v_{i+1}v_i) = 4m + 2$, then $S(b)$ holds. Similarly, if $c^*(f) = 4m + 1$, either $x, z \in V(G)$ and $S(a)$ holds or $x, z \in V(H)$ and $S(b)$ holds.

Finally, we consider $c^*(e) = 3$ and $c^*(f) = 4m$. If $x, z \in V(G)$, then there is an edge $tx$ such that $c^*(tx) = 2$ and $t \neq z$, so $S(a)$ holds. Otherwise, $x, z \in V(H)$, and there is an edge $zt$ such that $c^*(zt) = 4m + 1$ and $t \neq x$, so $S(b)$ holds. □

Thus we conclude the following.

**Corollary 9.** For all $n \geq 13$ and $n \equiv 1 \pmod{4}$, $\chi_\varepsilon(K_n) = n + 1$. 

2.3 The case \( n \equiv 3 \pmod{4} \)

Say \( n = 4m + 3, \ m \geq 1, \) and \( V(K_n) = \{v_0, \ldots, v_{4m+2}\}. \) Let \( G \) and \( H \) be the subgraphs of \( K_n \) induced by \( \{v_0, \ldots, v_{2m}\} \) and \( \{v_{2m+1}, \ldots, v_{4m+2}\} \), respectively. Then \( G \cong K_{2m+1}, \ H \cong K_{2m+2}, \) and each of them is \((2m+1)\)-edge colourable. We describe a colouring \( c \) of \( K_n \) in the colours \( 1, \ldots, 4m+5 \) as follows.

- In \( G, \) let \( c \) be any proper edge colouring of \( K_{2m+1} \) in the \( 2m+1 \) colours \( \{1, 2\} \cup \{m+4, \ldots, 3m+2\}. \)
- In \( H, \) let \( c \) be any proper edge colouring of \( K_{2m+2} \) in the \( 2m+1 \) colours \( \{4m+4, 4m+5\} \cup \{m+4, \ldots, 3m+2\}. \)
- We still need to colour the edges of the complete bipartite graph \( F \cong K_{2m+1,2m+2} \) induced by the edges \( v_iv_j, \ i \in \{0, \ldots, 2m\}, \ j \in \{2m+1, \ldots, 4m+2\}. \) But \( \chi(K_{2m+1,2m+2}) = 2m + 2 \) and there are \( 2m + 2 \) unused colours \( 3, \ldots, m+3 \) and \( 3m + 3, \ldots, 4m + 3. \) Colour the edges of \( F \) with these colours such that the following conditions are satisfied:
  
  - Let \( v_i \in V(G) \) be the vertex incident with no edge labelled 2. If \( v_j \in V(G) \) such that \( c(v_iv_j) = 1 \) and \( v_k \in V(H) \) such that \( c(v_iv_k) = 3, \) then \( c(v_jv_k) \neq 4m + 3. \)
  - Let \( v_p \in V(G) \) be the vertex incident with no edge labelled 1. If \( v_q \in V(G) \) such that \( c(v_pv_q) = 2 \) and \( v_r \in V(H) \) such that \( c(v_pv_r) = 3, \) then \( c(v_qv_r) \neq 4m + 3. \)

Such a colouring is easily found by arbitrarily assigning a proper colouring to \( F, \) and switching two colour classes if one of the two conditions is violated (there are at least four colour classes in \( F \) as \( m \geq 1).\)

As an example, a colouring of \( K_7 \) is given in Figure 3. It is clear that \( c \) is a proper edge colouring of \( K_{4m+3} \) in \( 4m + 5 \) colours.

**Theorem 10.** For all \( m \geq 1, \) the colouring \( c \) of \( K_{4m+3} \) has flatness equal to three.

**Proof.** Let \( F, \ G, \) and \( H \) be the subgraphs of \( K_{4m+3} \) defined above and let \( e, f \in E(K_{4m+3}) \) be adjacent edges such that \( c(e) < c(f). \) By Fact 4, it is sufficient to show \( S(a) \) or \( S(b) \) holds. Let \( e = v_jv_i \) and \( f = v_jv_k. \) Observe that at each vertex, exactly two colours do not appear as colours of edges incident with it, at least one of which is either 1 or \( 4m + 5. \) Thus if \( c(e) \geq 5, \) at least two of 2, 3, 4 are incident with \( v_j, \) and thus \( v_j \) is adjacent to some vertex \( v_l \neq v_k \) such that \( c(v_lv_j) < c(e), \) so \( S(a) \) holds.
Figure 3: Edge colouring $c$ of $K_7$

Similarly, if $c(f) \leq 4m + 1$, at least two of $4m + 2, 4m + 3, 4m + 4$ are incident with $v_k$, and thus $v_k$ is adjacent to some vertex $v_i \neq v_j$, such that $c(f) < c(v_kv_i)$, so $S(b)$ holds. Thus we consider $c(e) \in \{1, 2, 3, 4\}$ and $c(f) \in \{4m + 2, 4m + 3, 4m + 4, 4m + 5\}$.

Suppose $c(e) \in \{1, 2\}$. By construction, $c(f) \notin \{4m + 4, 4m + 5\}$. Thus $c(f) \in \{4m + 2, 4m + 3\}$, so $e \in E(G)$ and $f \in E(F)$. Thus there exists an edge $v_kv_i$ such that $c(v_kv_i) \in \{4m + 4, 4m + 5\}$. As $j \neq l$, $S(b)$ holds. Similarly, if $c(f) \in \{4m + 4, 4m + 5\}$, $c(e) \in \{3, 4\}$, so $f \in E(H)$ and $e \in E(F)$. Thus there exists an edge $v_lv_j$ such that $c(v_lv_j) \in \{1, 2\}$. As $j \neq l$, $S(a)$ holds.

Now we consider $c(e) \in \{3, 4\}$ and $c(f) \in \{4m + 2, 4m + 3\}$, and thus $e, f \in E(F)$. If $c(e) = 4$, then either there exists an edge $v_kv_i \in E(F)$ such that $c(v_kv_i) = 3$ and $l \neq k$ as $F$ is bipartite, so $S(a)$ holds; otherwise $v_j$ and $v_k$ are both in $V(H)$, and there exists a vertex $v_p, p \neq j$, such that $c(v_kv_p) \in \{4m + 4, 4m + 5\}$ and $S(b)$ holds. Similarly, if $c(f) = 4m + 2$, then either there exists an edge $v_kv_i \in E(F)$ such that $c(v_kv_i) = 4m + 3$ and $l \neq k$ as $F$ is bipartite, so $S(b)$ holds; otherwise $v_j$ and $v_k$ are both in $V(H)$, and there exists a vertex $v_p, p \neq j$, such that $c(v_kv_p) \in \{4m + 4, 4m + 5\}$ and $S(b)$ holds.

Finally, we consider $c(e) = 3$ and $c(f) = 4m + 3$. If $v_i \in V(G)$, then
there exists an edge $v_kv_l \in E(H)$ such that $c(v_kv_l) \in \{4m + 4, 4m + 5\}$ and $l \neq j$, so $S(b)$ holds. If $v_i \in V(H)$, then there exists an edge $v_i v_j \in E(G)$ such that $c(v_i v_j) \in \{1, 2\}$ and $l \neq k$ by construction, so $S(a)$ holds.

Thus we conclude the following.

**Corollary 11.** For all $n \geq 7$ and $n \equiv 3 \pmod{4}$, $\chi_\varepsilon(K_n) = n + 2$.

### 3 Conclusion

From Corollaries 6, 9, and 11, together with previous results, we obtain exact values of $\chi_\varepsilon$ for all $n \geq 6$:

**Theorem 12.** If $n \geq 6$, then

$$\chi_\varepsilon(K_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod{4} \\ n + 1 & \text{if } n \equiv 1, 2 \pmod{4} \\ n + 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

### Acknowledgements

The authors would like to thank C. M. Mynhardt for her valuable feedback during the preparation of this paper, as well as the anonymous referees for their helpful comments.

### References


Figure 4: Edge colouring $c^*$ of $K_{13}$ with flatness three.
Figure 5: Edge colouring of $K_9$ with flatness three.